Some new Travelling Wave Solutions for the Dissipative Compound Korteweg-de Vries Burgers Equation

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Abstract In this paper the effect of a small dissipation on waves is considered to find the exact solution to the dissipative Compound Kortewegde Vries Burgers equation (DCKdVB) in the presence of viscosity and in the absence of viscosity. With the help of the canonical form of Abel equation it is proved that the DCKdVB equation is integrable in terms of Weierstrass' elliptic functions.

Keywords: KdV-Burger equation, Traveling wave solutions, Elliptic solutions, Solitons, Wierstrass' elliptic function

1   Introduction

In this paper we consider the well-known compound Korteweg-de Vries-Burger (CKdVB) equation.

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = 0, \tag{1.1}$$

where p, q, r, s are all real constants. This equation is actually a combination of the KdV, mKdV and Burger equations, involving nonlinear dispersion and dissipation effects. For the different choices for p, q, r, s we can find different well-established differential equations.

For $p \neq 0$, $q \neq 0$, $r = 0$, $s \neq 0$; the (1.1) reduces the compound KdV equation as

$$u_t + puu_x + qu^2u_x - su_{xxx} = 0, \tag{1.2}$$

For $p = 0$, $q \neq 0$, $r \neq 0$, $s \neq 0$; the (1.1) reduces the mKdV-Burgers equation as

$$u_t + qu^2u_x + ru_{xx} - su_{xxx} = 0, \tag{1.3}$$

For $p \neq 0$, $q = 0$, $r \neq 0$, $s \neq 0$; the (1.1) reduces the KdV-Burgers equation as

$$u_t + puu_x + ru_{xx} - su_{xxx} = 0, \tag{1.4}$$

For $r = 0$ in (1.3), (1.4), we get the mKdV equation as

$$u_t + qu^2 u_x - su_{xxx} = 0, \tag{1.5}$$

and KdV equation as

$$u_t + puu_x - su_{xxx} = 0, \tag{1.6}$$

N. A. Kudryashov [16] worked on KdV (1.6) KdV-Burgers equations (1.4) and established some new travelling wave solutions of these equations. Zhaosheng Feng [5] obtained a new class of solutions of the Kortewegde VriesBurgers equation(1.4). Zheng et. al [17] have also obtained some new Exact Traveling Wave Solutions for Compound KdV-Burgers Equations (1.1) in Mathematical Physics. Stefan C. Mancas, Greg Spradlin, and Harihar Khanal [12] have solved for Weierstrass traveling wave solutions for dissipative Benjamin, Bona, and Mahony(BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = \nu u_{xx}$$

where ν is a transformed kinematic viscosity coefficient. A generalization to the equation(1.1) by adding a dispersion term is provided by the equation

$$u_t + puu_x + qu^2 u_x + ru_{xx} - su_{xxx} = \gamma u_x, \tag{1.7}$$

where γ is a real constant parameter, describes the unidirectional propagation of shallow water waves over a flat bottom (see[11]). The equation (1.7) is named as dissipative Compound Kortewegde Vries Burgers equation (DCKdVB).

2 General solutions of the DCKdVB equation in the travelling wave

We first present the general solution of the DCKdVB equation by using the travelling wave in Eq. (1.7). $u(x; t) = u(\zeta); \quad \zeta = x - ct$ is the travelling wave variables and $c \neq 0$ is the translational variable.
Then by considering $' = \frac{d}{d\zeta}$ (1.7) becomes
$-cu' + puu' + qu^2 u' + ru'' - su''' = \gamma u'$
Integrating and simplifying,

$$u'' - \frac{r}{s}u' + \frac{1}{s}\left[(\gamma + c)u - \frac{p}{2}u^2 - \frac{q}{3}u^3 + A\right] = 0, \tag{2.1}$$

where A is an integrating constant. To solve (2.1) we first state and prove the following lemma and using this lemma we will establish the solutions for the equation (2.1) in the form of travelling wave.

Lemma 1 *Solutions to a general second order ODE of the type*

$$u_{\zeta\zeta} + q_o(u) + q_2(u) = 0 \tag{2.2}$$

can be establish via the solutions to first kind of Abel's equation

$$\frac{dy}{du} = q_0(u)y^2 + q_2(u)y^3, \tag{2.3}$$

and vice versa by using the relationship $u_\zeta = \eta(u(\zeta))$.

Proof Since $u_\zeta = \eta(u(\zeta))$, we have $u_{\zeta\zeta} = \frac{d\eta}{du}u_\zeta = \eta\frac{d\eta}{du}$.

Then (2.2) reduces to the second kind of Abel's equation

$$\eta\frac{d\eta}{du} + q_0(u)\eta + q_2(u) = 0, \tag{2.4}$$

Let us take a transformation of the dependent variable as $\eta(u) = \frac{1}{y(u)}$. So the equation (2.4) becomes

$$\frac{dy}{du} = q_0(u)y^2 + q_2(u)y^3,$$

which is (2.3). By comparing (2.1) and (2.2),

$$q_0(u) = -\frac{r}{s}, \quad q_2(u) = \frac{1}{s}[(\gamma+c)u - \frac{p}{2}u^2 - \frac{q}{3}u^3 + A], \tag{2.5}$$

Again let us consider the linear transformation, $v = \int q_0(u)du = -\frac{r}{s}u$. Then (2.3) transformed into

$$\frac{dy}{dv} = y^2 + g(v)y^3, \tag{2.5a}$$

where $g(v) = -\frac{(\gamma+c)s}{r^2} + \frac{p}{2r^3}s^2v^2 - \frac{q}{3r^4}s^3v^3 - \frac{A}{r}$ is known as Appell's invariant. Now the Lemke's transformation,

$$y = -\frac{1}{z}\frac{dz}{dv}, \tag{2.5b}$$

transformed the equation (2.5a) into a second-order differential equation

$$z^2\frac{d^2v}{dz^2} + g(v) = 0. \tag{2.5c}$$

2.1   No viscosity, $r = 0$

In this case the equation (2.3) becomes $\frac{dy}{du} = q_2(u)y^3$, which reduces in the form by using (2.5)

$y^{-3}dy = \frac{1}{s}[(c+\gamma)u - \frac{p}{2}u^2 - \frac{q}{3}u^3 + A]du$

Integrating and using the relations $\eta(u) = \frac{1}{y(u)}$ and $u_\zeta = \eta(u(\zeta))$, we have

$$u_\zeta^2 = \frac{1}{s}[-(c+\gamma)u^2 + \frac{p}{3}u^3 + \frac{q}{6}u^4 - 2Au] - 2D, \tag{2.1.1}$$

where D is an integrating constant. This equation can be written as

$$\left(\frac{du}{d\zeta}\right)^2 = \frac{q}{6s}u^4 + \frac{p}{3s}u^3 - \frac{(\gamma+c)}{s}u^2 - \frac{2A}{s}u - 2D \equiv R(u), \text{say} \tag{2.1.2}$$

Let us consider the equation

$$\left(\frac{df}{dz}\right)^2 = a_0f^4 + 4a_1f^3 + 6a_2f^2 + 4a_3f + a_4 \equiv R(f).$$

This equation can be solved in terms of Weierstrass elliptic function $\wp$ (see[15]) as

$$f(z) = f_0 + \frac{R'(f_0)}{4[\wp(z;g_2,g_3) - \frac{1}{24}R''(f_0)]}$$

where prime denotes the derivative w.r.t f and f_0 is a simple root of $R(f)$. The invariants g_2, g_3 of $\wp(z;g_2,g_3)$ are given by

$g_2 = a_0a_1 - 4a_1a_3 + 3a_2^2$

$g_3 = a_0a_1a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4$ and the discriminant is $\triangle = g_2^3 - 27g_3^2$.
Thus the solution of (2.1.2) is,

$$u(\zeta) = u_0 + \frac{R'(u_0)}{4[\wp(\zeta; g_2, g_3) - \frac{1}{24}R''(u_0)]}$$

where u_0 is a simple root of $R(u)$. Invariants are
$g_2 = -\frac{qD}{3s} + \frac{pA}{6s^2} + \frac{c^2}{12s^2}$
$g_3 = \frac{qc}{18s^2} + \frac{pAc}{72s^3} + \frac{c^3}{216s^3} - \frac{qA^2}{24s^3} - \frac{p^2D}{72s^2}$
discriminant, $\triangle = g_2^3 - 27g_3^2$.
When $q = 0$, (2.1.1) reduces to

$$u_\zeta^2 = \frac{1}{s}[-(c+\gamma)u^2 + \frac{p}{3}u^3 - 2Au] - 2D, \tag{2.1.3}$$

Let us consider the special class of solutions that travel with a critical speed $c = -\gamma$.
If $c = -\gamma$ and $A = 0$, then the equation (2.1.2) becomes

$$\left(\frac{du}{d\zeta}\right)^2 = \frac{p}{3s}u^3 - 2D = \frac{p}{3s}\left(\frac{u^3}{3} + \frac{B^3}{3}\right), \text{where } D = \frac{B^3p}{6s}, \ B \neq 0 \tag{2.1.4}$$

Now let us take the transformation $\psi^2 = B + u$, where $\quad = \psi(\zeta)$. Then (2.1.4) becomes

$$4\psi^2\psi_\zeta^2 = \frac{p}{3s}\left(\frac{(\psi^2 - B)^3}{3} + \frac{B^3}{3}\right) \text{ or, } \psi_\zeta^2 = \frac{p}{12s}\left(\psi^4 - 3\psi^2 B + 3B^2\right) \tag{2.1.5}$$

Again we assume that $\psi = 3^{\frac{1}{4}}\sqrt{B}z(\zeta)$. Then (2.1.5) becomes

$$z_\zeta = \pm\frac{\sqrt{pB}}{3^{\frac{1}{4}}2\sqrt{s}}\sqrt{z^4 - \sqrt{3}z^2 + 1} \tag{2.1.6}$$

The ordinary differential equation (2.1.6) is solved by using Jacobi elliptic functions (see[12]).
Using Jacobi elliptic function, (2.1.6) gives

$\frac{2z}{1+z^2} = \pm Sn\left(\frac{\sqrt{pB}}{3^{\frac{1}{4}}\sqrt{s}}\zeta, k\right)$, where $k = \frac{\sqrt{3}+1}{2\sqrt{2}}$ = modulus of Jacobi elliptic function.
or, $\frac{2\psi}{3^{\frac{1}{4}}\sqrt{B}\left(1+\frac{\psi^2}{\sqrt{3}B}\right)} = \pm Sn\left(\frac{\sqrt{pB}}{3^{\frac{1}{4}}\sqrt{s}}\zeta, k\right)$
or, $\frac{2a\psi}{a^2+\psi^2} = \pm Sn\left(a\frac{\sqrt{p}}{\sqrt{3s}}\zeta, k\right)$, where $a = 3^{\frac{1}{4}}\sqrt{B}$
By solving the above quadratic equation we have,

$$= \pm a\frac{1 \pm Cn\left(a\frac{\sqrt{p}}{\sqrt{3s}}\zeta, k\right)}{Sn\left(a\frac{\sqrt{p}}{\sqrt{3s}}\zeta, k\right)} \tag{2.1.7}$$

Hence, the solution to the equation (1.7) without viscosity is

$$u(x,t) = \psi^2 - B = a^2\frac{\left[1\pm Cn\left(a\frac{\sqrt{p}}{\sqrt{3s}}\zeta, k\right)\right]^2}{Sn^2\left(a\frac{\sqrt{p}}{\sqrt{3s}}\zeta, k\right)} - B$$

$$= B\sqrt{3}\left[\frac{1 \mp Cn\left(\frac{\sqrt{pB}}{3^{\frac{1}{4}}\sqrt{s}}(x+\gamma t), k\right)}{1 \pm Cn\left(\frac{\sqrt{pB}}{3^{\frac{1}{4}}\sqrt{s}}(x+\gamma t), k\right)} - \frac{1}{\sqrt{3}}\right] \tag{2.1.8}$$
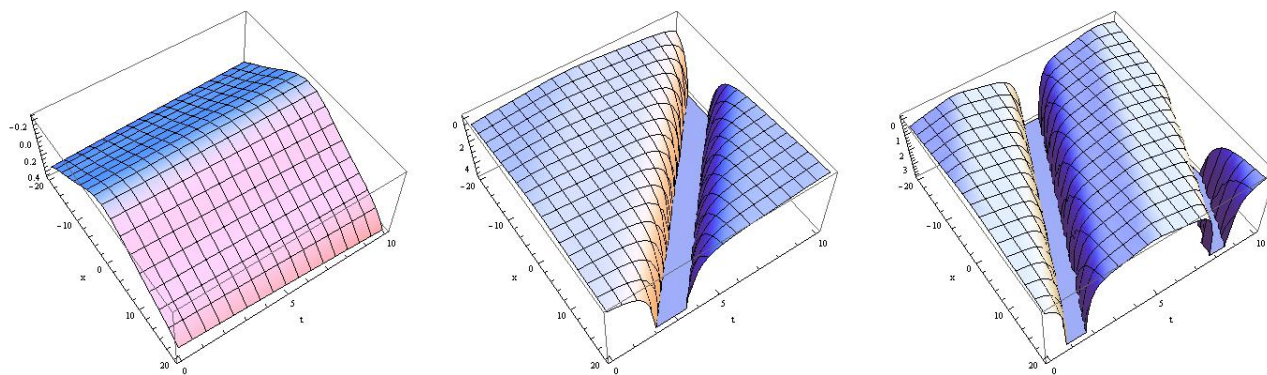
Fig.1. Traveling waves $r = 0$ with $B = 0.25$, $p = 0.25$, $s = 1$, left $\gamma = 0.2$, middle $\gamma = 4.5$ and right $\gamma = 9.5$, Eq. (2.1.8).

2.2 Viscosity present, $r > 0$

The equation (2.5a) is in the form of non-autonomous ordinary differential equation $F(y, y_v, v) = 0$. Since $v(z)$ has only poles of order two, then the solution of the equation (2.5c) must be obtained in terms of Weierstrass $\wp$ functions (see[14], p-431) via the transformation

$$v = E z^p w(z) + F \tag{2.2.1}$$

By substituting (2.2.1) in the equation (2.5c) with E, F constants, then for $p = \frac{2}{5}$, $\omega(z)$ will satisfy the elliptic equation

$$\omega^2 = 4\omega^3 - g_2\omega - g_3 \tag{2.2.2}$$

where g_2, g_3 are invariants. Since $\frac{dv}{dz} = -\frac{1}{yz}$ and $v(\zeta) = -\frac{r}{s}u(\zeta)$, we have

$$\frac{dz}{z} = \frac{r}{s}d\zeta \tag{2.2.3}$$

The Weierstrass $\wp$ solutions can be written into the combined general substitution

$$u(\zeta) = \alpha - g(\zeta) = \alpha - e^{-m\zeta}\Phi(\zeta) \tag{2.2.4}$$

where α and m are constants related to A and p. Substituting (2.2.4) into the equation (2.1),

$$g'' - \frac{r}{s}g' - \frac{q}{3s}g^3 + \left(\frac{p}{2s} + \frac{q\alpha}{s}\right)g^2 + \left(\frac{\gamma + c}{s} - \frac{p\alpha}{s} - \frac{q\alpha^2}{s}\right)g = 0 \tag{2.2.5}$$

here we consider

$$A = -(\gamma + c)\alpha + \frac{p}{2}\alpha^2 + \frac{q}{3}\alpha^3 \tag{2.2.6}$$

From (2.2.4) we have $g(\zeta) = e^{-m\zeta}\Phi(\zeta)$, therefore $g'(\zeta) = e^{-m\zeta}(\Phi' - m\Phi)$ and $g''(\zeta) = e^{-m\zeta}(\Phi'' - 2m\Phi' + m^2\Phi)$. Hence (2.2.5) becomes

$$\Phi'' - \left(2m + \frac{r}{s}\right)\Phi' + \left(m^2 + \frac{rm}{s} + \frac{\gamma + c}{s} - \frac{p\alpha}{s} - \frac{q\alpha^2}{s}\right)\Phi = \frac{q}{3s}e^{-2m\zeta}\Phi^3 - \left(\frac{p}{2s} + \frac{q\alpha}{s}\right)e^{-m\zeta}\Phi^2 \tag{2.2.7}$$

We now consider $\Phi(\zeta) = w(z(\zeta))$. By using the chain rule with $' = \frac{d}{d\zeta}$ and $* = \frac{d}{dz}$ we have
$\Phi'(\zeta) = w^*z'$ and
$\Phi''(\zeta) = w^{**}z'^2 + w^*z''$

Therefore the equation (2.2.7) becomes

$$z'^2 w^{**} + \left[z'' - \left(2m + \frac{r}{s} \right) z' \right] w^* + \left(m^2 + \frac{rm}{s} + \frac{\gamma + c}{s} - \frac{p\alpha}{s} - \frac{q\alpha^2}{s} \right) w = \frac{q}{3s} e^{-2m\zeta} w^3 - \left(\frac{p}{2s} + \frac{q\alpha}{s} \right) e^{-m\zeta} w^2$$

(2.2.8)

By setting

$$z'' - \left(2m + \frac{r}{s} \right) z' = 0$$

(2.2.9)

we get

$$z' = c_1 e^{\left(2m + \frac{r}{s} \right)\zeta}$$

(2.2.10)

where c_1 is an arbitrary constant. Again if we choose

$$m^2 + \frac{rm}{s} + \frac{\gamma + c}{s} - \frac{p\alpha}{s} - \frac{q\alpha^2}{s} = 0$$

(2.2.11)

then the equation (2.2.8) reduces to the following simplify equation

$$z'^2 w^{**} - \frac{q}{3s} e^{-2m\zeta} w^3 + \left(\frac{p}{2s} + \frac{q\alpha}{s} \right) e^{-m\zeta} w^2 = 0$$

(2.2.12)

The equation (2.2.12) can be make a more simple by choosing $q = 0$ and it reduces to

$$z'^2 w^{**} + \frac{p}{2s} e^{-m\zeta} w^2 = 0$$

(2.2.13)

Our aim is now to solve the differential equation (2.2.13) with the help of (2.2.10) by setting $m = -\frac{2}{5}\frac{r}{s}$. Therefore the equation (2.2.13) becomes

$$c_1^2 e^{\frac{2}{5}\frac{r}{s}\zeta} w^{**} = -\frac{p}{2s} e^{\frac{2}{5}\frac{r}{s}\zeta} w^2$$

$$\text{or, } w^{**} = -\frac{p}{2sc_1^2} w^2 = 6w^2 \text{ where } c_1^2 = -\frac{p}{12s}$$

Integrating above equation we obtain

$$w^{*2} = 4w^3 - g_3$$

(2.2.14)

where g_3 is an integrating constant. Solution to the differential equation (2.2.14) is given by in terms of Weierstrass $\wp$ function as

$$w(z) = \wp(z + c_3, 0, g_3)$$

(2.2.15)

with invariants $g_2 = 0$ and g_3. Here c_3 is an arbitrary constant.

Now the solution to the differential equation (2.2.10) is

$$z(\zeta) = c_2 + \frac{5c_1 s}{r} e^{\frac{r}{5s}\zeta}$$

(2.2.16)

where c_2 is an integrating constant. Hence we have with $c_4 = c_2 + c_3$

$$\Phi(\zeta) = w(z(\zeta)) = \wp\left(c_4 + \frac{5c_1 s}{r} e^{\frac{r}{5s}\zeta}, 0, g_3 \right)$$

(2.2.17)

Since here $q = 0$ and $m = -\frac{2}{5}\frac{r}{s}$ the equation (2.2.11) becomes $\alpha = \frac{1}{p}\left[\gamma + c - \frac{6}{25}\frac{r^2}{s} \right]$ and (2.2.6) reduces to $A = -(\gamma + c)\alpha + \frac{p}{2}\alpha^2$.

Hence we have

$$\alpha = \frac{\gamma + c \pm \sqrt{(\gamma + c)^2 + 2pA}}{p}$$

(2.2.18)

Therefore the general solution to the differential equation (2.2.2) is

$$u(\zeta) = \alpha - g(\zeta) = \frac{\gamma + c \pm \sqrt{(\gamma + c)^2 + 2pA}}{p} - e^{\frac{2r}{5s}\zeta} \wp\left(c_4 + \frac{5c_1 s}{r} e^{\frac{r}{5s}\zeta}, 0, g_3 \right)$$

(2.2.19)

By setting a particular value for the velocity c and the other parameters, we obtain α and A and substitution of these values in the equation (2.2.19) gives the solution of the differential equation (1.7).

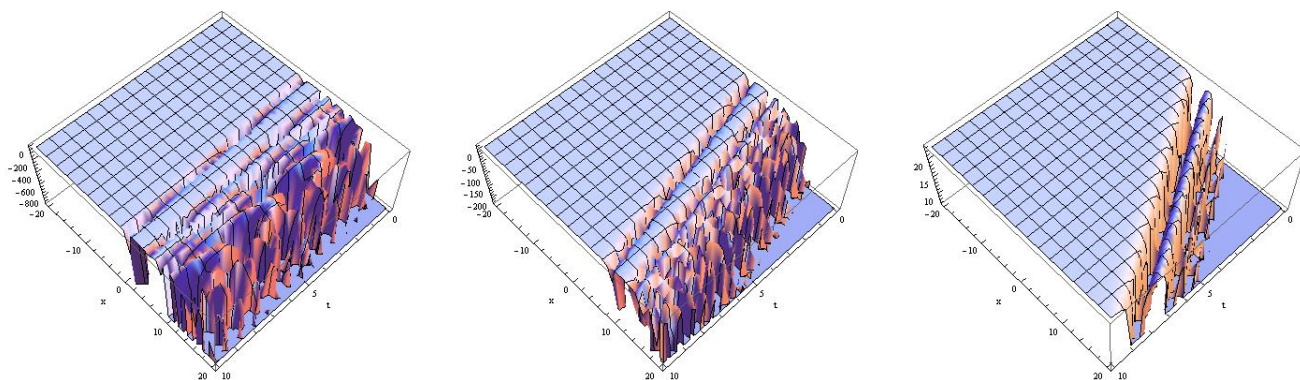
Fig.2. Solution for $r = 1$ with $A = 1$, $p = 0.25$, $s = 1$ $\gamma = 0.5$, $c_1 = 0.5$, $c_4 = 1$, $g_3 = 1$, left $c = 0.25$, middle $c = 1.0$ and right $c = 2.4$, Eq. (2.2.19).
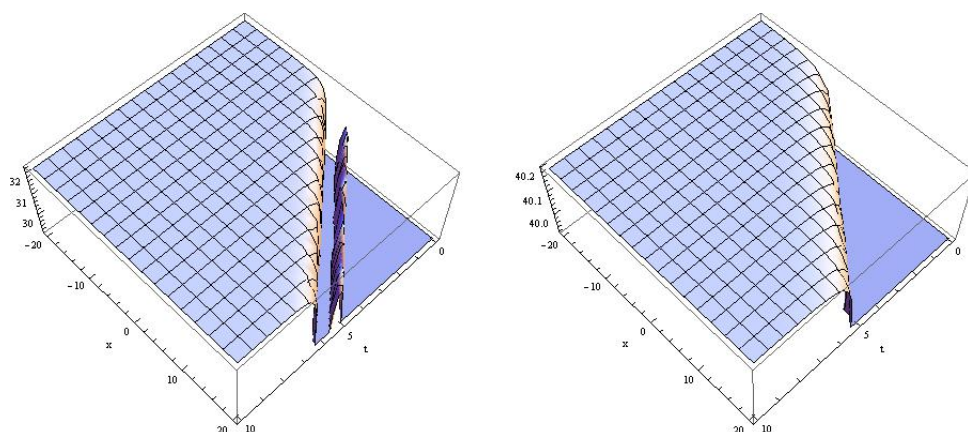

Fig.3. Solution for $r = 1$ with $A = 1$, $p = 0.25$, $s = 1$ $\gamma = 0.5$, $c_1 = 0.5$, $c_4 = 1$, $g_3 = 1$, left $c = 3.5$ and right $c = 4.5$, Eq. (2.2.19).

3   Conclusions

In this paper we considered Compound Kortewegde Vries Burgers equation(CKdVB)and introduced the dissipation term and finally we got the dissipative Compound Kortewegde Vries Burgers equation (DCKdVB). With the help of the canonical form of Abel equation it is proved that the DCKdVB equation is integrable in terms of Weierstrass' elliptic functions. We solved this equation in two ways (i) when the viscous term is not present(i.e, $r = 0$) and it has traveling wave solutions which depend critically on the traveling wave velocity. (ii) When the viscous term is present(i.e, $r > 0$) and we solved it with the help of Weierstrass' elliptic functions.

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